THE CAUCHY PROBLEM FOR THE POROUS MEDIUM EQUATION WITH VARIABLE DENSITY. ASYMPTOTIC BEHAVIOR OF SOLUTIONS

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We study the intermediate asymptotics of non-negative solutions to the Cauchy problem

$$(\mathbf{P}) \qquad \begin{cases} \rho(x)\,\partial_t u = \Delta u^m & \text{in } \ Q := \mathbb{R}^n \times \mathbb{R}_+ \\ u(x,\,0) = u_0 \end{cases}$$

in dimensions $n \geq 3$. We assume that $m > 1$ (slow diffusion) and $\rho(x)$ is positive, bounded and behaves as a negative power, $\rho(x) \sim |x|^{-\gamma}$ as $|x| \to \infty$, $\gamma > 0$. The data u_0 are assumed to be nonnegative and such that $\int_{\mathbb{R}^n} \rho(x) u_0\, dx < \infty$ (finite energy). The presented results are contained in **[RV]** and **[KRV]**.

The answer depends on whether $0 < \gamma < 2$ or $\gamma > 2$. In the first case, the asymptotic behavior is described in terms of a one-parameter family of source-type solutions $U_E(x,t) = t^{-\alpha} F_E(xt^{-\beta})$, $E > 0$, of the related singular problem

$$\begin{cases} |x|^{-\gamma} u_t = \Delta u^m & \text{in } \ Q \\ |x|^{-\gamma} u(x,\ 0) = E\delta(x) \end{cases}$$

Thus, this case represents a natural extension of the corresponding result for the standard PME ($\gamma = 0$).

The behaviour for $\gamma > 2$ strongly departs from the previous one. Indeed, in this case, solutions to **(P)** have a *universal* long-time behavior in separated variables of the form

$$u(x,t) \sim t^{-1/(m-1)} W(x),$$

where $V = W^m$ is the unique bounded, positive solution of the sublinear elliptic equation $-\Delta V = c\,\rho(x) V^{1/m}$ in $\mathbb{R}^n$ vanishing as $|x| \to \infty$; $c = 1/(m-1)$. Such a behavior of u is typical of Dirichlet problems on bounded domains with zero boundary data.

If $\rho(x)$ has an intermediate decay, $\rho \sim |x|^{-\gamma}$ with $2 < \gamma < \gamma_2 := N - (N-2)/m$, solutions still enjoy the finite propagation property (as in the case of lower γ). In this range a more precise description may be given at the diffusive scale in terms of source-type solutions $U(x,t)$ of the related singular equation $|x|^{-\gamma} u_t = \Delta u^m$. Thus in this range we have *two* different space-time scales in which the behavior of solutions is non-trivial. The corresponding results complement each other and agree in the intermediate region where both apply, thus providing an example of matched asymptotics.

References.

[RV] G. Reyes & J. L. Vázquez, *Long time behaviour for the inhomogeneous PME in a medium with slowly decaying density*, Commun. Pure Appl. Anal. **8** (2) (2009), pp. 493–508.

[KRV] S. Kamin, G. Reyes & J. L. Vázquez, *Long time behavior for the inhomogeneous PME in a medium with rapidly decaying density*, Discrete and Continuous Dynamical Systems, (special issue devoted to Nonlinear Parabolic Problems), DCDS-A **26** (2) (2010), pp. 521–549.