Combinatorics Seminar

Wednesday, October 16th, 2024

4:00-5:00 pm in Hume 321

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Phase transitions in isoperimetric problems

Abstract

Among all bodies of a fixed volume, the ball has the smallest surface area. Among all ideals in $R[x_1, ..., x_n]$ generated by a fixed number of monomials of degree d, the one with the fewest monomials of degree $d+1$ is the lex-ideal.

These are two very different sounding facts, but are in fact both examples of isoperimetric problems, which ask: among all objects of a particular size, which has the smallest boundary? In this talk, we'll focus on the discrete setting, where isoperimetric problems take the form: given a graph G, among all subsets of the vertices of a fixed size, which has the smallest neighborhood? In general, this is a very hard question to answer, but every so often, there is a beautiful solution.

Two of the most amazing solutions arise from nice grids on $\mathbb{Z}^n$, namely the ℓ_∞ and ℓ_1 grids. In both cases, there is a well-ordering of $\mathbb{Z}^n$ such that initial segments are always optimal for the isoperimetric problem! Motivated by these examples and other evidence, Barber–Erde recently asked if this phenomenon continues for any Cayley graph on $\mathbb{Z}^n$.

We show that the answer is "no", even in the case of $n = 1$ by constructing Cayley graphs whose isoperimetric problem undergoes an abrupt "phase transition". The construction is motivated by the realization that one can mimick small-scale higher-dimensional phenomena in low dimensions.

Time permitting, we will also discuss more recent work on the isoperimetric problem on the bounded ℓ_∞ grid, which we can show exhibits many phase transitions, and introduce a new notion which we dub "iso-complexity". (Joint with with Joe Briggs) (Time permitting: joint with Ben Baker, Joe Briggs and Manuel Fernandez)